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STEADY-STATE RESPONSE IN THE PARAMETER PLANE*

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Summary: This paper considers the adjustment of parameters in linear control systems to meet both steady-state and transient response specifications. A graphical method using parameter plane techniques is presented whereby the steady-state error may be minimized while being constrained by the transient response requirements. Another method is proposed whereby, for a specified steady-state response, the transient response is given in evidence as a function of three system parameters. The procedures are illustrated by examples for both continuous and sampled-data multi-loop control systems.

The problem of linear control system design can be interpreted as the adjustment of the system parameters to meet both steady-state and transient response requirements. The Šiljak parameter plane method[1-3] gives in evidence the pole-zero locations of a system closed-loop transfer function in terms of two system parameters. In the application of the method, the transient response requirements are translated into constraints imposed on the pole-zero configuration of the system transfer function. The steady-state response on the other hand, can be specified by the error constants. As shown in this paper, the error constants are functions of adjustable system parameters and may be introduced into the parameter plane. This enables the steady-state and transient response to be considered simultaneously in a control system design.

Two graphical design procedures are presented. In the first, error constant contours are plotted on the parameter plane enabling the steady-state error to be minimized subject to constraints imposed on the system pole-zero locations. In the second procedure, the transient response is studied on the parameter plane for a specified error constant, and is given in evidence as a function of three system parameters.

The parameter plane curves are readily plotted using a digital computer, and so the methods are presented as a practical approach to the design of linear control systems. In the following sections, the parameter plane approach and the error constant definitions are reviewed for reference, the two design procedures are developed, and examples illustrating the procedures are discussed for both continuous and sampled-data multi-loop control systems.

SUMMARY OF THE PARAMETER PLANE METHOD

The characteristic equation considered in the design of linear continuous systems is

$$f(s) = \sum_{k=0}^{m} a_k s^k = 0. \tag{1}$$

The complex roots of equation 1 are expressed in terms of their damping ratio ζ and their natural frequency ω_n

$$s = -\omega_n\zeta \pm j\,\omega_n\sqrt{1-\zeta^2} \tag{2}$$

Substituting equation 2 into equation 1 results in two equations which may be manipulated and conveniently expressed in terms of the Chebyshev functions as follows

$$\begin{aligned} \sum_{k=0}^{m} (-1)^k a_k \omega_n^k U_k(\zeta) &= 0 \\ \sum_{k=0}^{m} (-1)^k a_k \omega_n^k U_{k-1}(\zeta) &= 0 \end{aligned} \tag{3}$$

The Chebyshev functions $U_k(\zeta)$ may be calculated from the recurrance relationship

$$U_{k+1}(\zeta) - 2\zeta\, U_k(\zeta) + U_{k-1}(\zeta) = 0 \tag{4}$$

where $U_o(\zeta) \equiv 0$ and $U_1(\zeta) \equiv 1$.

The coefficients of the characteristic equation are expressed as

$$a_k = b_k\, \alpha + c_k\, \beta + h_k\, \alpha\beta + d_k \tag{5}$$

where b_k, c_k, h_k, and d_k are functions of the fixed parameters of the system and are therefore of known numerical value, and α and β are the two system parameters to be selected in the design.

The substitution of equation 5 into equation 3 results in two equations

$$\begin{aligned} B_1\, \alpha + C_1\, \beta + H_1\, \alpha\beta + D_1 &= 0 \\ B_2\, \alpha + C_2\, \beta + H_2\, \alpha\beta + D_2 &= 0 \end{aligned} \tag{6}$$

where

$$\begin{aligned} B_1 &= \sum_{k=0}^{m} (-1)^k\, b_k\, \omega_n^{\,k}\, U_{k-1}(\zeta), & B_2 &= \sum_{k=0}^{m} (-1)^k\, b_k\, \omega_n^{\,k}\, U_k(\zeta) \\ C_1 &= \sum_{k=0}^{m} (-1)^k\, c_k\, \omega_n^{\,k}\, U_{k-1}(\zeta), & C_2 &= \sum_{k=0}^{m} (-1)^k\, c_k\, \omega_n^{\,k}\, U_k(\zeta) \\ H_1 &= \sum_{k=0}^{m} (-1)^k\, h_k\, \omega_n^{\,k}\, U_{k-1}(\zeta), & H_2 &= \sum_{k=0}^{m} (-1)^k\, h_k\, \omega_n^{\,k}\, U_k(\zeta) \\ D_1 &= \sum_{k=0}^{m} (-1)^k\, d_k\, \omega_n^{\,k}\, U_{k-1}(\zeta), & D_2 &= \sum_{k=0}^{m} (-1)^k\, d_k\, \omega_n^{\,k}\, U_k(\zeta) \end{aligned} \tag{7}$$

By solving equations 6 for α and β, the ζ and ω_n curves are plotted in the $\alpha\beta$-plane. The complex roots of equation 1 are then determined by interpolating between these curves.

When the roots of the characteristic equation are real, i.e.,

$s = -\sigma$, equation 1 may be expressed as

$$\alpha \sum_{k=o}^{m} b_k(-\sigma)^k + \beta \sum_{k=o}^{m} c_k(-\sigma)^k + \alpha\beta \sum_{k=o}^{m} h_k(-\sigma)^k + \sum_{k=o}^{m} d_k(-\sigma)^k = 0 \qquad (8)$$

Constant σ lines are readily plotted in the parameter plane using equation 8. Therefore, both real and complex roots are given in evidence as functions of α and β.

For sampled-data systems, z-transform theory is applied and the characteristic equation is expressed as follows

$$f(z) = \sum_{k=o}^{m} a_k z^k = 0 \qquad (9)$$

where

$$z = \varepsilon^{sT}, \qquad (10)$$

T being the sampling period.

The complex roots of $f(z) = 0$ are expressed in terms of the z-plane damping ratio ζ_z and the z-plane natural frequency ω_z.

$$z = \omega_z \angle \cos^{-1}(\zeta\ ) = \omega_z \zeta_z \pm j\, \omega_z \sqrt{1 - \zeta_z^2}\ . \qquad (11)$$

The real roots of $f(z)$ are given by

$$z = -\sigma_z\ . \qquad (12)$$

The form of these equations is similar to that given for continuous systems. Equations 6, 7, and 8 may therefore be used to map the parameter plane in terms of ζ and ω_n for a sampled-data system by making the following substitutions derived from equations 2, 10, 11 and 12.

$$\begin{aligned} &\exp(-\omega_n \zeta T) \quad \text{for} \quad \omega_n \\ &-\cos(\omega_n T \sqrt{1 - \zeta^2} \quad \text{for} \quad \zeta \\ &\exp(-\sigma T) \quad \text{for} \quad \sigma\ . \end{aligned} \qquad (13)$$

The only portion of the s-plane considered is the primary strip; i.e., $|\omega \sqrt{1 - \zeta^2}| \leq \pi/T$.

ERROR CONSTANTS

The symbol K_e will be used to represent either the position error constant K_p, the velocity error constant K_v, or the acceleration constant K_a. The error constants are defined for the continuous system and sampled-data system as follows.

For Continuous Systems	For Sampled-Data Systems
$K_p = \lim_{s \to o} G(s)$	$K_p = \lim_{z \to 1} G(z)$
$K_v = \lim_{s \to o} s\, G(s)$	$K_v = \lim_{z \to 1} (1-z)\, G(z)/T$
$K_a = \lim_{s \to o} s^2\, G(s)$	$K_a = \lim_{z \to 1} (1-z)^2\, G(z)/T^2$ (14)

where $G(s)$ and $G(z)$ are the system open-loop transfer functions. The steady-state errors for a unit step function, ramp or parabolic input to the system are given by $1/(1+K_p)$, $1/K_v$, $1/K_a$, respectively.

The next section considers the minimization of a system steady-state error subject to the transient response specifications. The analytical maximization of K_e subject to relative stability constraints is considered and then the graphical maximization of K_e on the parameter plane subject to the transient response specifications is introduced and illustrated by examples.

MINIMIZATION OF STEADY-STATE ERROR ON THE $\alpha\beta$-PLANE

Consider the error constant K_e, expressed as a function of the two system parameters α and β.

$$K_e = K_e(\alpha, \beta) \tag{15}$$

The solutions of the simultaneous equations

$$\frac{\partial K_e}{\partial \alpha} = 0 \quad \text{and} \quad \frac{\partial K_e}{\partial \beta} = 0 \tag{16}$$

which satisfy the conditions.

$$\frac{\partial^2 K_e}{\partial \alpha^2} < 0, \quad \left|\frac{\partial^2 K_e}{\partial \alpha \partial \beta}\right| - \frac{\partial^2 K_e}{\partial \alpha^2} \cdot \frac{\partial^2 K_e}{\partial \beta^2} < 0 \qquad (17)$$

are the maxima values of K_e on the $\alpha\beta$-plane. The absolute maximum K_e is therefore readily determined. However, the transient response associated with this value of K_e is usually not acceptable. To ensure a more acceptable transient response, relative stability constraints may be included into the analytical derivation as follows.

Consider the maximization of K_e subject to the constraints that all the roots of the characteristic equation have damping ratios ζ greater than or equal to the prescribed value ζ_o; i.e., $\zeta \geq \zeta_o$. The region of interest in the $\alpha\beta$-plane is therefore bounded by the $\zeta = \zeta_o$ curve. If the maximum K_e for this region occurs inside the boundary, then its value may be determined from equations 16 and 17. If the maximum K_e for the region does not occur inside the boundary, then it occurs on the boundary itself and its value may be calculated from the following equation

$$\left(\frac{\partial K_e}{\partial \omega_n}\right)_{\zeta=\zeta_o} = \left(\frac{\partial K_e}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial \omega_n} + \frac{\partial K_e}{\partial \beta} \cdot \frac{\partial \beta}{\partial \omega_n}\right)_{\zeta=\zeta_o} = 0 \qquad (18)$$

and the condition $\left(\frac{\partial^2 K_e}{\partial \omega_n^2}\right)_{\zeta=\zeta_o} < 0$.

The partial derivatives $\frac{\partial \alpha}{\partial \omega_n}$ and $\frac{\partial \beta}{\partial \omega_n}$ may be obtained from equations 6 and 7 in a straightforward manner, but the calculation of the maxima values of K_e may become tedious as the roots of a high degree algebraic equation in ω_n must be found and examined.

The constraints $\zeta \geq \zeta_o$ are not sufficient to ensure an acceptable transient response, but to include further constraints analytically would be difficult. Using the parameter plane, however, the problem of adjusting the system parameters to give the best compromise between steady-state and transient response may be attempted by plotting contours of constant K_e on the $\alpha\beta$-plane. The curves are interpreted as in the following examples.

Consider the system of Fig. 1 where

$$G_1(s) = K_1 \frac{(s+1)}{(s+0.2)}, \qquad G_2(s) = \frac{K_2}{s(s^2+0.8s+1)}$$

$$G_{-1}(s) = K_{-1}\, s \qquad , \qquad G_{-2} = K_{-2}\, s^2 .$$

The system characteristic equation is

$$s^4 + (\alpha+1)s^3 + (0.2\alpha+\beta+1.16)s^2 + (0.2\beta+\gamma+0.2)s + \gamma = 0 \qquad (19)$$

where $\alpha = K_2K_{-2}$ and $\beta = K_2K_{-1}$ and $\gamma = K_1K_2$.

The error constant K_v can be expressed as

$$K_v = \frac{\gamma}{0.2(1+\beta)} . \qquad (20)$$

Fig. 2 gives the parameter plane curves for the condition $\gamma = 20$. By plotting just a few ζ lines, the region of interest in the $\alpha\beta$-plane is evident. It is seen that perhaps the best compromise between the steady-state and the transient response for a specified ζ_o is not at the point of maximum K_v given by analytical methods (e.g., M_1 for $\zeta_o = 0.5$; M_2 for $\zeta_o = 0.3$), but in the region where the real axis characteristic roots are conveniently located (e.g., M_3 for $\zeta_o = 0.3$).

In a system design, the transient response may be calculated at points of interest in the parameter plane. The roots of the characteristic equation for these points may be found approximately by interpolation from the parameter plane or, more accurately, by an iterative method. In the example, the characteristic equation roots M_1, M_2, M_3 are given as follows.

M_1	M_2	M_3
$\zeta_1 = 0.5$	$\zeta_1 = 0.3$	$\zeta_1 = 0.3$
$\omega_{n_1} = 2.25$	$\omega_{n_1} = 2.25$	$\omega_{n_1} = 1.25$
$\zeta_2 = 0.69$	$\zeta_2 = 0.86$	$\sigma_1 = 1.7$
$\omega_{n_2} = 1.99$	$\omega_{n_2} = 1.98$	$\sigma_2 = 7.4$

For the regions M_1 and M_2, the two pairs of complex poles are now given explicitly, and it is seen that for the regions M_1 and M_2 the transient response is unacceptable. The response for the region M_1 may be calculated using Laplace transform methods and a design compromise achieved. Further, the $\alpha\beta$-plane curves may be replotted for different specified parameters in order to explore the possibilities of an improved design. Usually, when a time constant is considered as the one parameter α, and a gain as the other parameter β, $\alpha\beta$-product terms appear in the characteristic equation and the results of reference 2 are used to plot the parameter plane.

Consider as a second example the sampled-data system given in Fig. 3[3] where the sampling period T is one second and

$$G_1(s) = \frac{1 - e^{-Ts}}{s}, \qquad G_2(s) = \frac{K_2}{s + 0.19}$$

$$G_4(s) = \frac{0.741\ K_4}{s(s+3.9)}, \qquad G_{-1}(s) = \frac{1.9K_{-1}\ s}{s + 1.9},$$

the z-transforms of interest are

$$G_1G_2(z) = K_2 \frac{0.91}{z - 0.827}, \qquad G_3G_4(z) = K_4 \frac{0.142z + 0.186}{(z-1)(z-0.02)}$$

$$G_3G_4G_{-1}(z) = K_4K_{-1} \frac{0.139z + 0.019}{(z - 0.15)(z - 0.02)}.$$

The characteristic equation may be written as follows.

$$f(z) = z^4 + (0.189\alpha - 2)z^3 + (-0.237\alpha + 0.092\beta + 1.14)z^2$$
$$+ (0.081\alpha + 0.014\beta - 0.146)z + (0.01\alpha - 0.004\beta - 0.002) = 0 \quad (21)$$

where $\alpha = K_4 K_{-1}$ and $\beta = K_2 K_4$.

The error constant K_v is given from the definition 15 as

$$K_v = \frac{1.763\beta}{1 + 0.19\alpha} \quad . \quad (22)$$

The maximum K_v for a specified settling time is given in evidence in Fig. 4. For a settling time $\omega_n\zeta = 0.1$; i.e., $\omega_z = 0.905$, the maximum K_v occurs at the point M in Fig. 4 (i.e., $K_v = 2.7$). It is to be noted that the steady-state errors are given only at the sampling instants.

The parameter plane in these examples may be plotted with the error constant as one of the axis. All that is required is that the equation for K_v be incorporated into the computer program which calculates the various parameter plane curves. The parameter plane method is not limited to the damping ratio (ζ) and the settling time ($\omega_n\zeta$) contours as illustrated in the above examples, but any convenient contour may be plotted and therefore other stability constraints may be considered in a system design.

The error constant for a system may be specified, in which case the transient response may be investigated as a function of α, β, and a third parameter γ on a single parameter plane diagram. This case is now considered.

THREE-PARAMETER PROBLEM

If the error constant K_e is specified, the parameter plane method may be extended to study the characteristic equation root locations as a function of three parameters. The effect of adjusting the two parameters α and β may be studied on the $\alpha\beta$-plane, and for each point a third parameter γ is then given as a known function of α, β, and K_e. An example from the preceeding section is used to illustrate the design procedure. For the system of Fig. 1, if K_v is specified, the characteristic equation may be expressed in terms of α and β by eliminating the third parameter γ from equations 19 and 20. For $K_v = 10$, $\gamma = 2(1+\beta)$ and

$$s + (\alpha + 1)s + (0.2\alpha + \beta + 1.16)s + (2.2\beta + 2.2)s + (2\beta + 2) = 0 \quad (23)$$

From the parameter plane plot of equation 23 in Fig. 5, it is seen that α and β may be selected to give a good transient response and γ may then be calculated for trial points and the system designed (e.g., for M, $\zeta = 0.5$, $\omega_n = 2.5$, $\sigma_1 = 1.5$, $\sigma_2 = 2.1$). This approach to the simultaneous consideration of steady-state and transient response enables the designer to examine readily the effectiveness of various compensating networks. All that is required is the interpretation of the parameter plane diagrams corresponding to each proposed system.

CONCLUSIONS

It has been shown that the steady-state and transient response may be considered simultaneously in a linear control system design using the parameter plane. Two procedures were developed and illustrated by examples of both continuous and sampled-data multi-loop control systems. In the first procedure, the steady-state error was minimized subject to the transient response requirements; while in the second procedure, the effects of three system parameters on the transient response were given in evidence for a specified error constant.

The scope of the problem discussed in this paper is but one aspect of the more general problem of introducing specifications and constraints into the analytical or graphical design procedures in order to give in evidence more relevant information for the system design. The work of this paper may be extended to include more directly the transient response specifications into the design procedures such that the effect of more system parameters may be considered.

REFERENCES

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3. D.D. Šiljak, Analysis and Synthesis of Feedback Control Systems in the Parameter Plane - Part II, Sampled-Data Systems, IEEE Trans. on Applications and Industry, Vol. 83, No. 75, Nov. 1964; pp 458-466.

CAPTIONS

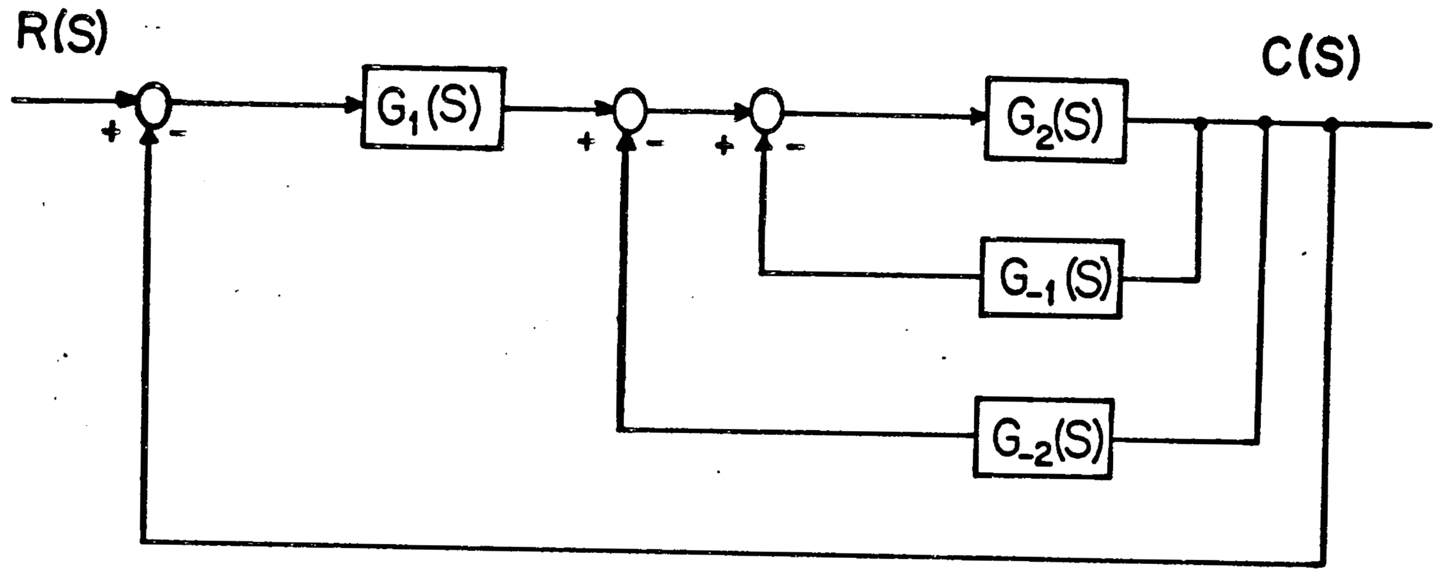
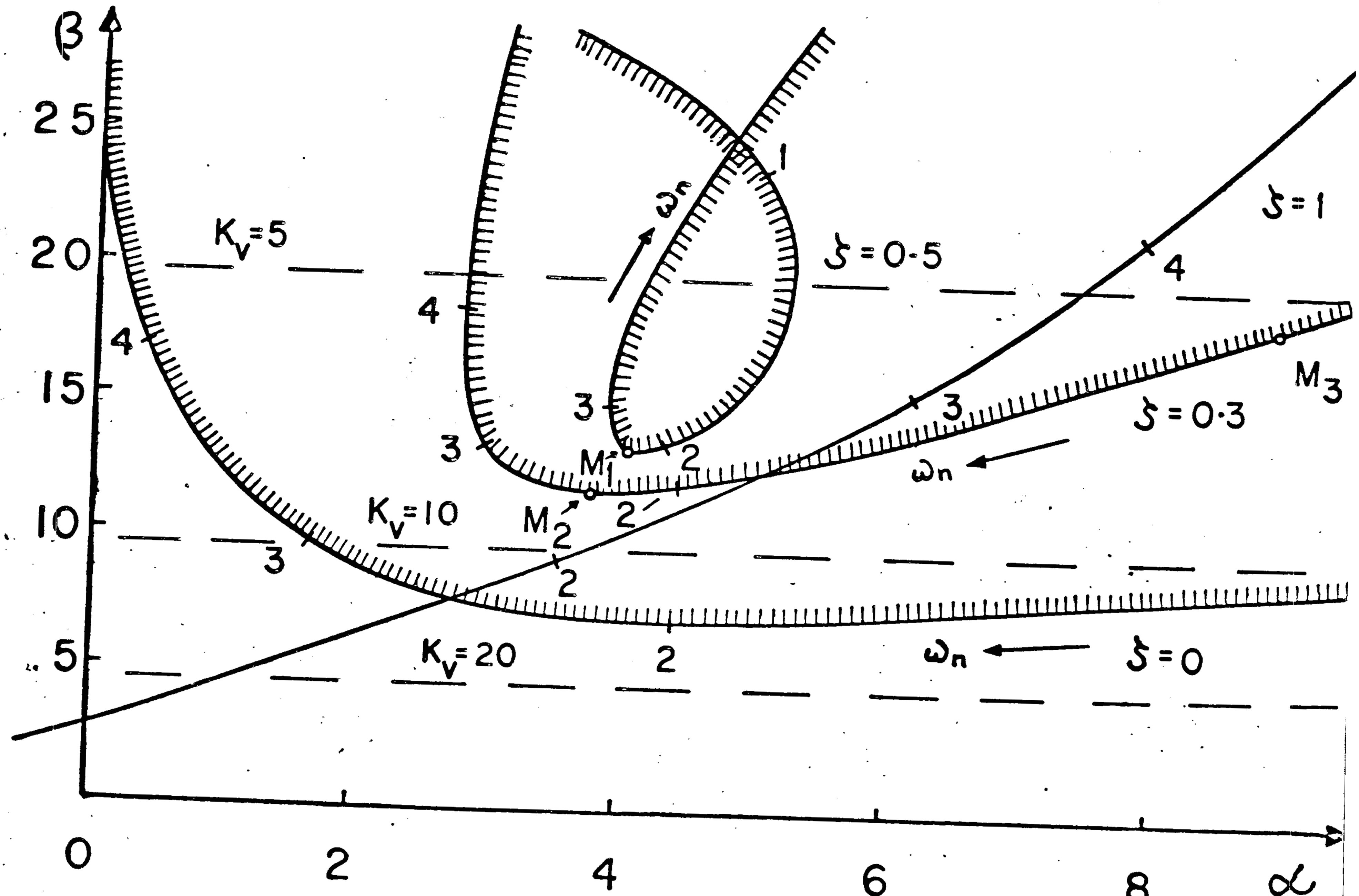


Fig 1



β
α
0
2
4
6
8
5
10
15
20
25
$K_v=5$
$K_v=10$
$K_v=20$
ζ=1
ζ=0·5
ζ=0·3
ζ=0
ω_n
ω_n
ω_n
M_1
M_2
M_3

Fig 2

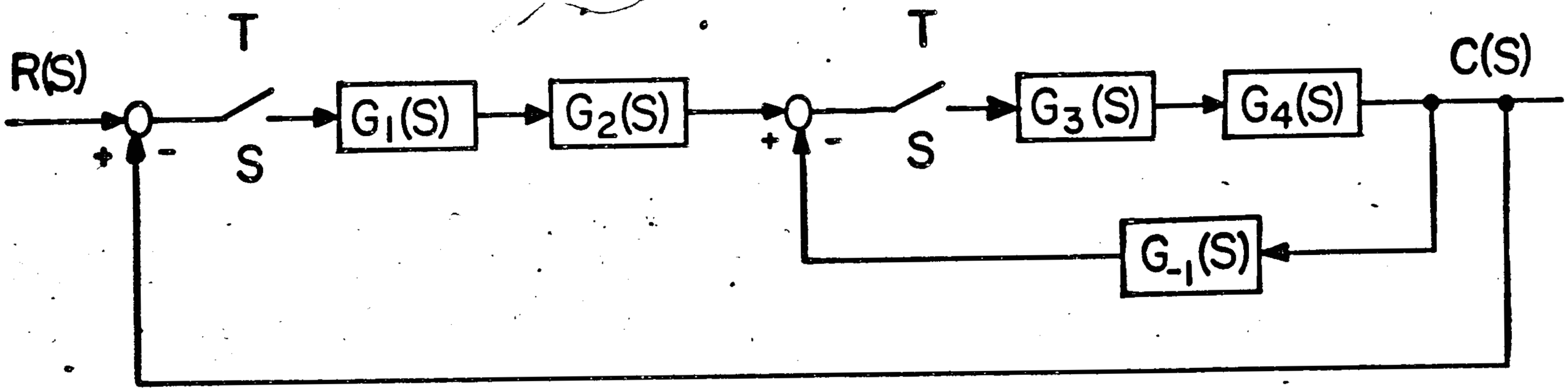
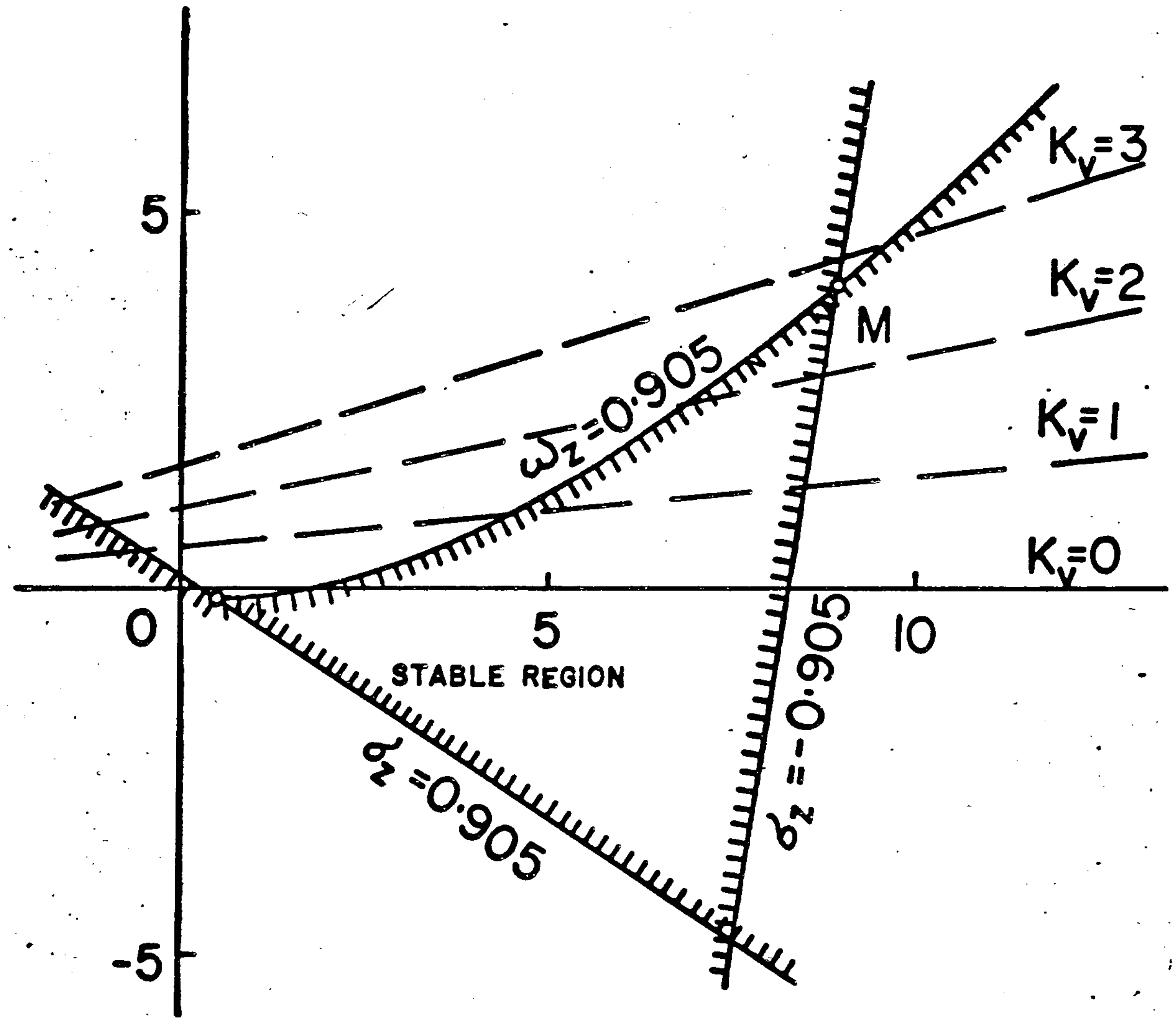


Fig 3.



5
-5
0
5
10
STABLE REGION
$K_v=3$
$K_v=2$
$K_v=1$
$K_v=0$
M
$\omega_z=0{\cdot}905$
$\sigma_z=0{\cdot}905$
$\delta_z=-0{\cdot}905$

Fig 4

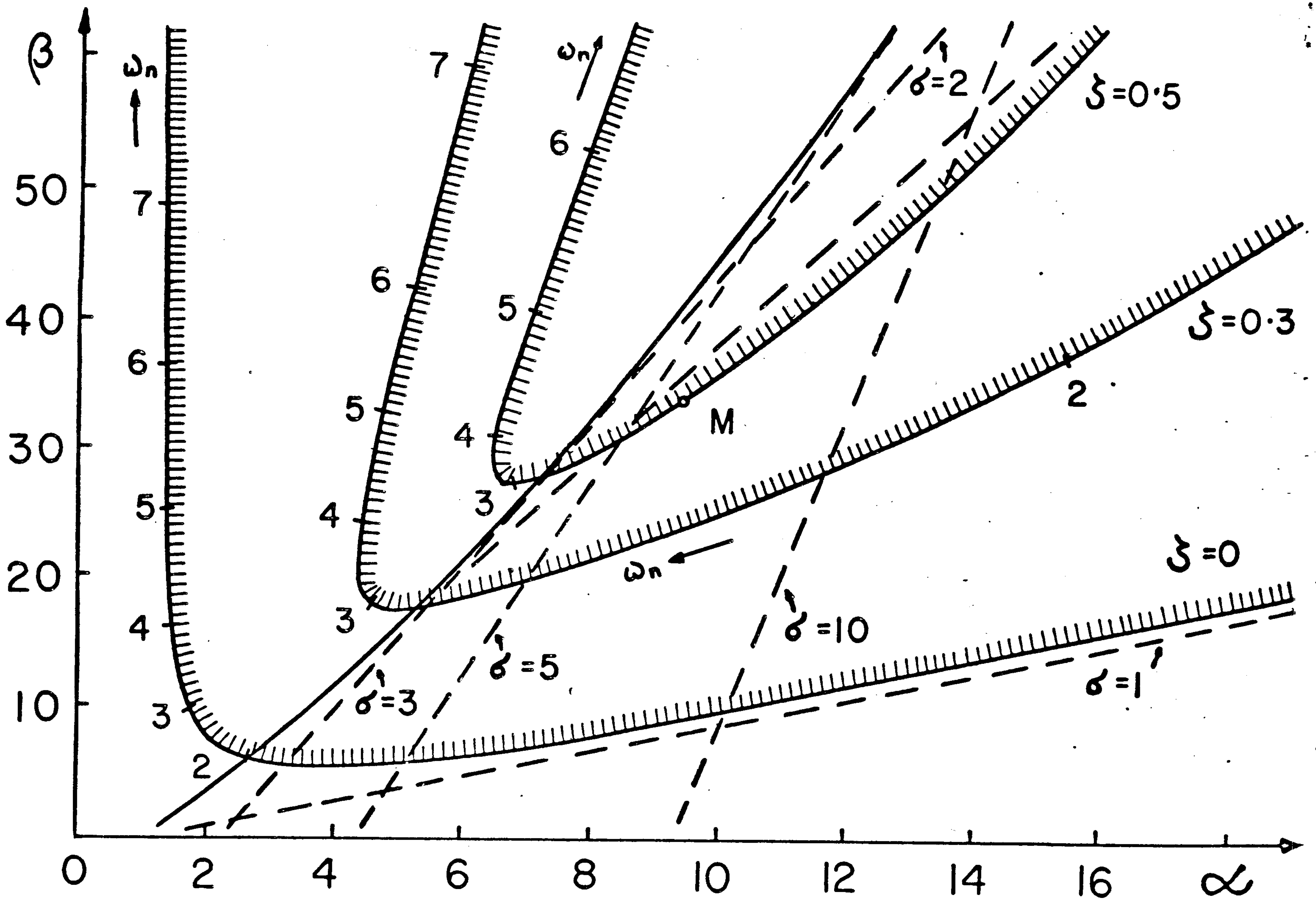


Fig 5